ON THE DEVELOPMENT OF WAVES ON THE FREE SURFACE AND ON THE SURFACE SEPARATING TWO FLUIDS UNDER THE ACTION OF TRAVELING PRESSURES

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This paper considers the spatial problem of unsteady waves which arise on the free surface and on the surface of separation between two fluids under the action of a periodic pressure system applied to the free surface in an infinite strip. This work is an extension of the work of [1-4] in which the steady and unsteady oscillations of a fluid surface, maintained by periodic perturbations, are considered.

1. Let a layer of fluid of density ρ_1 and depth h float on the surface of an infinitely deep fluid of density ρ. At the initial moment of time $t = 0$ both fluids are at rest and the free surface and the surface of separation are horizontal. We shall set ourselves the problem of investigating the form of the waves which arise on the free surface and on the surface of separation under the action of a periodic traveling pressure system of the form

$$p = p_0 \exp[i(kx - \omega t)] \qquad \text{for } |y| < a \tag{1.1}$$

applied at time $t = 0$ to the free surface.

We shall designate the velocity potentials of the lower and upper fluids by $\varphi(x, y, z, t)$ and $\varphi_1(x, y, z, t)$. Then, to determine these functions we have [3] the two equations

$$\Delta\varphi_1 = 0 \quad (0 < z < h), \qquad \Delta\varphi = 0 \quad (z < 0) \tag{1.2}$$

with the boundary conditions

$$\left(\frac{\partial^2\varphi_1}{\partial t^2} + g\frac{\partial\varphi_1}{\partial z}\right)_{z=} = \frac{1}{\rho_1}\frac{\partial p}{\partial t}, \qquad \frac{\partial\varphi}{\partial z} = \frac{\partial\varphi_1}{\partial z} \qquad \text{at } z = 0$$

$$\rho \frac{\partial^2 \varphi}{\partial t^2} - \rho_1 \frac{\partial^2 \varphi_1}{\partial t^2} + g(\rho - \rho_1) \frac{\partial \varphi}{\partial z} = 0 \qquad \text{at} \quad z = 0 \tag{1.3}$$

The forms of the free surface ζ_1 and the surface of separation ζ are given by the formulas

$$\zeta_1 = \frac{1}{g} \left(\frac{\partial \varphi_1}{\partial t}\right)_{z=h} - \frac{p}{\rho_1 g}, \qquad \zeta = \frac{1}{g(\rho - \rho_1)} \left(\rho \frac{\partial \varphi}{\partial t} - \rho_1 \frac{\partial \varphi_1}{\partial t}\right)_{z=0} \tag{1.4}$$

To the boundary conditions (1.3) it is necessary to add the initial conditions

$$\begin{aligned} &\frac{\partial \varphi_1}{\partial t} = \frac{p}{\rho_1} && \text{at} \quad z = 0,\ t = 0; && \varphi_1(x, y, z, 0) = 0 \\ &\rho \frac{\partial \varphi}{\partial t} = \rho_1 \frac{\partial \varphi_1}{\partial t} && \text{at} \quad z = 0,\ t = 0; && \varphi(x, y, z, 0) = 0 \end{aligned} \tag{1.5}$$

which express the fact that the free surface and the surface of separation are horizontal and both fluids are at rest at the initial moment of time. We shall introduce the pressure (1.1) in the form of a Fourier integral

$$p = p_0 e^{i(kx - \omega t)} \int_{-\infty}^{\infty} \frac{\sin ma}{m} e^{imy} dm$$

and we shall seek functions $\varphi_1(x, y, z, t)$ and $\varphi(x, y, z, t)$ in the following integral form:

$$\varphi_1 = \frac{p_0}{\pi \rho_1} e^{ikx} \int_{-\infty}^{\infty} \frac{\sin ma}{m} e^{imy} [A(m, t) e^{\Delta z} + B(m, t) e^{-\Delta z}] \, dm \tag{1.6}$$

$$\varphi = \frac{p_0}{\pi \rho} e^{ikx} \int_{-\infty}^{\infty} \frac{\sin ma}{m} e^{imy} C(m, t) e^{\Delta z} \, dm \qquad (\Delta = \sqrt{k^2 + m^2}) \tag{1.7}$$

The functions φ_1 and φ represented in the form of (1.6) and (1.7) satisfy Equations (1.2) for arbitrary values of the functions A, B, C. After satisfying conditions (1.3) and (1.5), we obtain the system

$$\ddot{A} e^{\Delta h} + \ddot{B} e^{-\Delta h} + g\Delta (A e^{\Delta h} - B e^{-\Delta h}) = -i\omega e^{-i\omega t}$$

$$\ddot{C} - \ddot{A} - \ddot{B} + g\Delta (1 - \gamma) C = 0, \qquad A - B = \gamma C \qquad (\gamma = \rho_1 / \rho)$$

with the initial conditions

$$\dot{A}(m, 0) e^{\Delta h} + \dot{B}(m, 0) e^{-\Delta h} = 1, \quad \dot{A} + \dot{B} - \dot{C} = 0, \quad A(m, 0) = B(m, 0) = C(m, 0) = 0$$

for determining A, B, C.

A dot indicates differentiation with respect to time. Hence, we have the system

$$\ddot{A}e^{\Delta h} + \ddot{B}e^{-\Delta h} + g\Delta\,(Ae^{\Delta h} - Be^{-\Delta h}) = -i\omega e^{-i\omega t}$$
$$\ddot{A} - \beta\ddot{B} + g\Delta\,(A - B) = 0 \qquad (\beta = 1 + \gamma\,/\,1 - \gamma) \tag{1.8}$$

with the initial conditions

$$A(m, 0) = B(m, 0) = 0, \qquad \dot{A}(m, 0) = \frac{\beta e^{-\Delta h}}{\beta + e^{-2\Delta h}}, \qquad \dot{B}(m, 0) = \frac{e^{-\Delta h}}{\beta + e^{-2\Delta h}} \tag{1.9}$$

Moreover, C is determined from the equation $C = \gamma^{-1}(A - B)$.

Solving the system (1.8) with the initial conditions (1.9), we find

$$A(m, t) = \sum_{k=0}^{4} a_k e^{in_k t}, \qquad B(m, t) = \sum_{k=0}^{4} b_k e^{in_k t}, \qquad C(m, t) = \sum_{k=0}^{4} \gamma^{-1}(a_k - b_k)\, e^{in_k t}$$

where

$$n_0 = -\omega, \qquad n_{1,2} = \pm\sqrt{g\Delta}, \qquad n_{3,4} = \pm\sqrt{\xi g\Delta}$$
$$D(m) = (1 + \gamma)\,\omega^2 + (1 - \gamma)\,[-g\Delta + (\omega^2 + g\Delta)\, e^{-2\Delta h}], \qquad b_1 = b_2 = 0$$
$$b_0 = \frac{i\omega(1 - \gamma)}{D(m)}\, e^{-\Delta h}, \qquad b_1 = 0, \qquad b_2 = 0, \qquad b_3 = \frac{b_0(\omega - n_3) - ib_5}{2n_3}$$
$$\xi = \frac{(1 - \gamma)\,[1 - e^{-2\Delta h}]}{1 + \gamma + (1 - \gamma)\, e^{-2\Delta h}}, \qquad b_4 = -\frac{b_0(\omega + n_3) - ib_5}{2n_3}$$
$$b_5 = \frac{(1 - \gamma)\, e^{-\Delta h}}{1 + \gamma + (1 - \gamma)\, e^{-2\Delta h}}, \qquad a_0 = \frac{-i\omega + (\omega^2 + g\Delta)\, b_0 e^{-\Delta h}}{g\Delta - \omega^2}\, e^{-\Delta h}$$
$$a_1 = \frac{1}{2k_1}\,[a_0(\omega - n_1) - qb_3(n_1 + n_3) - qb_4(n_1 - n_3) - ia_5]$$
$$a_2 = -\frac{1}{2k_1}\,[a_0(\omega + n_1) + qb_3(n_1 - n_3) + qb_4(n_1 + n_3) - ia_5]$$
$$a_3 = qb_3 e^{-2\Delta h}, \qquad a_4 = qb_4 e^{-2\Delta h}, \qquad a_5 = \frac{(1 + \gamma)\, e^{-\Delta h}}{1 + \gamma + (1 - \gamma)\, e^{-2\Delta h}}, \qquad q = \frac{1 + \xi}{1 - \xi}\, e^{-2\Delta h}$$

Substituting the derived values of the functions A, B, C into Formulas (1.4), we find

$$\zeta_1 = \frac{ip_0}{\pi p_1 g}\, e^{ikx} \int_{-\infty}^{\infty} \sum_{k=0}^{4} f_k e^{i(my + n_k t)}\, dm, \qquad \zeta = \frac{ip_0}{\pi \rho_1 g}\, e^{ikx} \int_{-\infty}^{\infty} \sum_{k=0}^{4} \psi_k e^{i(my + n_k t)}\, dm \tag{1.10}$$

where

$$f_k = n_k\,[a_k e^{\Delta h} + b_k e^{-\Delta h}]\, \frac{\sin ma}{m}, \qquad \psi_k = n_k\,[a_k - \beta b_k]\, \frac{\sin ma}{m} \tag{1.11}$$

The Formulas (1.10) which are obtained are analytical expressions of the form of the waves which arise on the free surface and on the surface of separation.

2. We shall undertake the investigation of the first of the Formulas (1.10) which determines the form of the waves on the free surface. To begin with we note that individual terms which enter into the sums under the integral have a pole of first order on the real axis. Thus f_0 has a pole of first order on the real axis at the points

$$m_{1,2} = \pm \sqrt{\sigma^2 - k^2} \qquad (\sigma = \omega^2 g^{-1} > k), \qquad m_{3,4} = \pm \sqrt{\alpha^2 - k^2} \tag{2.1}$$

where $\alpha > \beta\sigma$ is the only positive root of the equation

$$(1 + \gamma)\,\omega^2 + (1 - \gamma)\,[-g\alpha + (\omega^2 + g\alpha)\,e^{-2\alpha h}] = 0 \tag{2.2}$$

The function $f_2(m)$ has a pole of first order at the points m_1, m_2; $f_4(m)$ at the points m_3, m_4. The functions f_1, f_3 do not have singularities on the real axis. The singularities of the individual terms which enter into the sum under the integral in the first formula of (1.10) cancel, and therefore the function under the integral does not have singularities on the path of integration and the integration along the real axis can be replaced by integration along a contour L which goes around the poles m_1 and m_3 along small semi-circles lying in the lower half-plane and around the poles m_2 and m_4 along small semi-circles lying in the upper half-plane. Formula (1.10) can be written thus:

$$\zeta_1 = \frac{i p_0}{\pi \rho_1 g}\, e^{ikx} \sum_{k=0}^{4} J_k, \qquad J_k = \int_{(L)} f_k e^{i(my + n_k t)}\, dm \tag{2.3}$$

Considering the region $y > a$ (the region $y < -a$ is symmetrical), we obtain

$$J_0 = e^{-i\omega t} \Big[\int_{(c)} f_0 e^{imy}\, dm + 2\pi i \sum_{k=1,3} \operatorname{res}\,(f_0 e^{imy})_{m_k} \Big] \tag{2.4}$$

where the contour (c) goes around the poles m_1, m_2, m_3, m_4 in the upper half-plane. Carrying out the calculation of the residues and taking into consideration that the integral in Formula (2.4) is a quantity of order $(ky)^{-1}$ for large values of ky, we have

$$J_0 = -N_1 - N_2 + O\,[(ky)^{-1}] \tag{2.5}$$

$$N_1 = \frac{2\pi\omega^4 \sin a\sqrt{\sigma^2 - k^2}}{\omega^4 - k^2 g^2} \left(1 - \frac{2e^{-2\sigma h}}{\beta - 1 - 2e^{-2\sigma h}}\right) \exp\,[i(\sqrt{\sigma^2 - k^2}\,y - \omega t)]$$

$$N_2 = \frac{4\pi \sin a\sqrt{\alpha^2 - k^2}}{(\alpha^2 - k^2)(\alpha - \sigma)}\; \frac{\alpha^2 \sigma e^{-2\alpha h}}{1 + [2h(\alpha + \sigma) - 1]\, e^{-2\alpha h}} \exp\,[i(\sqrt{\alpha^2 - k^2}\,y - \omega t)]$$

We shall now consider J_2, writing it in the following form:

$$J_2 = \int\limits_{(L)} f_2 \exp\,[kyM_2(m)]\,dm, \qquad M_2(m) = i\,(m - \nu \sqrt[4]{1 + m^2}), \qquad \nu = \frac{t}{y}\left(\frac{g}{k}\right)^{1/2}$$

We shall carry out the investigation of J_2 for large values of ky by the method of stationary phase. The stationary points are roots of the equation $M_2'(m) = 0$.

It is evident that the location of the stationary points on the real axis essentially depends on the value of the parameter ν, i.e. on the ratio of t to y. At the same time the points m_1 and m_2, poles of the function $f_2(m)$, occupy fixed positions on the real axis. Investigating the position of the stationary points relative to the poles as a function of values of the parameter ν, we find that Re $M_2(m) \leqslant 0$ on the initial contour L for $y < u_1 t$ and on the contour L_1, which goes around the pole m_1 from above, for

$$y > u_1 t, \qquad u_1 = \frac{g\sqrt{\omega^4 - k^2 g^2}}{2\omega^3} \tag{2.6}$$

Therefore

$$J_2 = \begin{cases} N_1 + O\,[(ky)^{-1/2}] & (y > u_1 t) \\ O\,[(ky)^{-1/2}] & (y < u_1 t) \end{cases} \tag{2.7}$$

We shall pass on to the investigation of J_4, writing it as follows:

$$J_4 = \int\limits_{(L)} f_4 \exp\,[kyM_4(m)]\,dm, \qquad M_4(m) = i\,[m - \nu\sqrt{\xi\Delta}]$$

The function $f_4(m)$ has poles only at the points m_3 and m_4. Investigating the position of the stationary points of the expression $M_4(m)$ relative to their poles for various values of the parameter ν, we come to the conclusion that Re $M_4(m) \leqslant 0$ on the initial contour L for $y < u_2 t$ and on the contour L_2 which goes around the pole m_3 from above for $y > u_2 t$, where

$$u_2 = \frac{\omega\sqrt{\alpha^2 - k^2}}{2\alpha^2}\left[1 + \frac{2h\sigma(\alpha\sigma^{-1} + 1)}{1 - e^{-2\alpha h}}\, e^{-2\alpha h}\right] \tag{2.8}$$

Therefore

$$J_4 = \begin{cases} N_2 + O\,[(ky)^{-1/2}] & (y > u_2 t) \\ O\,[(ky)^{-1/2}] & (y < u_2 t) \end{cases} \tag{2.9}$$

Since $f_1(m)$ and $f_3(m)$ do not have poles on the real axis, for large values of ky we find

$$J_1 = O\,[(ky)^{-1/2}], \qquad J_3 = O\,[(ky)^{-1/2}] \tag{2.10}$$

Formulas (2.3), (2.5), (2.7), (2.9), (2.10) give the following final expression of the form of the waves on the free surface of a liquid:

$$\zeta_1 = \eta_1 + \eta_2 \tag{2.11}$$

where

$$\eta_1 = \begin{cases} \varkappa_1 \sin (kx + y\sqrt{\sigma^2 - k^2} - \omega t) & (y < u_1 t) \\ O\left[(ky)^{-1/2}\right] & (y > u_1 t) \end{cases} \tag{2.12}$$

$$\eta_2 = \begin{cases} \varkappa_2 \sin (kx + y\sqrt{\alpha^2 - k^2} - \omega t) & (y < u_2 t) \\ O\left[(ky)^{-1/2}\right] & (y > u_2 t) \end{cases} \tag{2.13}$$

$$\varkappa_1 = \frac{2p_0\omega^4 \sin a\sqrt{\sigma^2 - k^2}}{\rho_1 g(\omega^4 - k^2 g^2)}\left[1 - \frac{2e^{-2\sigma h}}{\beta - 1 + 2e^{-2\sigma h}}\right]$$
$$\varkappa_2 = -\frac{4p_0 \sin a\sqrt{\alpha^2 - k^2}}{\rho_1 g(\alpha^2 - k^2)(\alpha - \sigma)} \, \frac{\alpha^2\sigma e^{-2\alpha h}}{1 + [2h(\alpha + \sigma) - 1]e^{-2\alpha h}} \tag{2.14}$$

We shall pass on to the second expression of (1.10). Since the poles of the functions ψ_k and f_k coincide and the whole sum under the integral in (1.10) has no singularities on the real axis, we then obtain, after replacing integration along the real axis with integration along the contour L

$$\zeta = \frac{ip_0}{\pi\rho_1 g} e^{ikx} \sum_{k=0}^{4} J_k, \quad J_k = \int_{(L)} \psi_k e^{i(n_k t + my)} dm \tag{2.15}$$

Carrying out the investigation of the integrals J_k in an analogous manner to the investigation of the integrals J_k, we find the following final expression for the wave profiles on the surface of separation:

$$\zeta = \eta_3 + \eta_4 \tag{2.16}$$

where

$$\eta_3 = \begin{cases} \varkappa_3 \sin (kx + y\sqrt{\sigma^2 - k^2} - \omega t) & (y < u_1 t) \\ O\left[(ky)^{-1/2}\right] & (y > u_1 t) \end{cases} \tag{2.17}$$

$$\eta_4 = \begin{cases} \varkappa_4 \sin (kx + y\sqrt{\alpha^2 - k^2} - \omega t) & (y < u_2 t) \\ O\left[(ky)^{-1/2}\right] & (y > u_2 t) \end{cases} \tag{2.18}$$

$$\varkappa_3 = e^{-h\sigma}\varkappa_1, \qquad \varkappa_4 = -(1 - \gamma)^{-1}\varkappa_2 \tag{2.19}$$

From Formulas (2.11) and (2.16) it follows that in the case under consideration two systems of progressive waves of the form (2.12) and (2.13) arise on the free surface. Moreover, the foreward front of the waves of (2.12) travel along the y-axis with velocity u_1 and the foreward front of the waves of (2.13) with velocity u_2, where u_1 and u_2 are equal respectively to the projections on the y-axis of the group velocities of the waves of (2.12) and (2.13).

The waves which arise on the surface of separation differ only in amplitude from the waves which arise on the free surface. It turns out that the amplitude κ_3 of the waves (2.17) is smaller than the amplitude κ_1 of the waves on the free surface by the factor exp (σh) and the amplitude κ_4 of the waves (2.18) is larger than the amplitude of the waves (2.13) by the factor $(1 - \gamma)^{-1}$.

For values of β, h, σ which satisfy the inequality $\exp(-2\sigma\beta h) \ll 1$, the formulas for κ_1, κ_2, u_2 are considerably simplified and take the form ($\alpha = \beta\sigma$)

$$\varkappa_1 = \frac{2p_0\omega^4 \sin a\sqrt{\sigma^2 - k^2}}{\rho_1 g\,(\omega^4 - k^2 g^2)}, \qquad \varkappa_2 = \frac{4p_0\alpha^2\sigma \sin a\sqrt{\alpha^2 - k^2}}{\rho_1 g\,(\alpha^2 - k^2)(\alpha - \sigma)}\,e^{-2\alpha h}, \qquad u_2 = \frac{\omega\sqrt{\alpha^2 - k^2}}{2\alpha^2}$$

The amplitudes κ_3 and κ_4 are given by Formulas (2.19). From Formulas (2.6) and (2.8) we find

$$u_1 \gtrless u_2 \quad \text{for } \beta \gtrless \frac{k^2}{\sigma^2 - k^2}$$

3. The work, which corresponds to the pressure force (1.1) on the free surface in the strip $|y| < a$, per unit length of the pressure waves will be equal to

$$W = p_0 \int_x^\chi \cos(kx - \omega t)\left[\int_{-b}^{b}\left(\frac{\partial\varphi_1}{\partial z}\right)_{z=h} dy\right] dx \quad \left(\chi = x + \frac{2\pi}{k}\right) \tag{3.1}$$

$$\left(\frac{\partial\varphi_1}{\partial z}\right)_{z=h} = \frac{p_0}{\pi\rho_1} e^{ikx} \int_{(L)} \sum_{k=0}^{4} \vartheta_k e^{i(my + n_k t)}\, dm, \quad \vartheta_k = \Delta\,(a_k e^{\Delta h} - b_k e^{-\Delta h})\,\frac{\sin ma}{m} \tag{3.2}$$

at the moment of time t.

Here L is the initial path of integration. From Formulas (1.11) and (3.2) it is seen that the poles of the functions f_k and ϑ_k coincide. Taking into account that L Re $(in_k) \leqslant 0$ ($k = 1, \ldots, 4$) on the path of integration and that the derivative $n_k'(m)$ has only one stationary point $m = 0$, we find that Formula (3.2) takes the form

$$\left(\frac{\partial\varphi_1}{\partial z}\right)_{z=h} = \frac{p_0}{\pi\rho_1} e^{i(kx - \omega t)} \int_{(L)} \vartheta_0 e^{imy} dm + O\,[(\sqrt{gkt})^{-1/2}]$$

for large values of $\sqrt{(gkt)}$.

Substituting the latter equality into Formula (3.1) and carrying out the integration, we find the following expression for the work E of the applied pressures per unit length of the pressure waves over the period $\tau = 2\pi\omega^{-1}$ for large values of the time t($\sqrt{(gkt)} \gg 1$):

$$E = E_1 + E_2, \qquad E_1 = \frac{8\pi^2 p_0^2 (\beta - 1)\sigma^2 \sin^2 a\sqrt{\sigma^2 - k^2}}{k\rho_1 g(\sigma^2 - k^2)^{3/2}[\beta - 1 + 2e^{-2\Delta h}]}$$

$$E_2 = \frac{16\pi^2 p_0^2 \alpha^2 \sigma e^{-2\alpha h} \sin^2 a\sqrt{\alpha^2 - k^2}}{k\rho_1 g(\alpha^2 - k^2)^{3/2}(\alpha - \sigma)\{1 + [2h(\alpha + \sigma) - 1]e^{-2\Delta h}\}}$$

The expressions which we have found for undamped waves on the free surface and the surface of separation and the expression for the work of the applied pressures are valid for values of $\sigma > k$. In the case $\sigma < k$, $\alpha > k$ undamped waves of the form (2.12) and (2.17) will not arise; moreover, as follows from Formula (2.19), the amplitudes of waves on the surface of separation will be larger than the amplitudes of waves on the free surface by the factor $(1 - \gamma)^{-1}$. The very same thing will also occur in the case $\sigma > k$ when the equality

$$a\sqrt{\sigma^2 - k^2} = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

is satisfied.

In both of the cases considered the term E_1, which enters into the expression for the pressure work, will be absent. We note that undamped waves will not arise for values of $\alpha < k$ and that the applied pressures will not do work for large values of the time t.

By the very same method a solution is obtained to the planar problem of unsteady waves which arise on the free surface and the surface of separation under the action of periodic pressures of the form $p = p_0 \exp(-i\omega t)$ applied at the free surface at time $t = 0$ in the region $|x| < a$. The waves on the free surface ζ_1 and on the surface of separation ζ have the form

$$\zeta_1 = \eta_1 + \eta_2, \qquad \zeta = \eta_3 + \eta_4,$$

where

$$\eta_{1,3} = \begin{cases} \varkappa_{1,3} \sin(\sigma x - \omega t) & (y < u_1 t), \\ O[(\sigma x)^{-1/2}] & (y > u_1 t), \end{cases} \qquad \eta_{2,4} = \begin{cases} \varkappa_{2,4} \sin(\alpha x - \omega t) & (y < u_2 t) \\ O[(\sigma x)^{-1/2}] & (y > u_2 t) \end{cases}$$

Here the wave amplitudes and the velocities u_1, u_2 are given by Formulas (2.14), (2.19), (2.6), (2.8) if k is set equal to zero in them. In the case $a\sigma = n\pi (n = 0, \pm 1, \pm 2, \ldots)$ the amplitude of the waves which arise on the surface of separation will be larger than the amplitude of the waves on the free surface by the factor $(1 - \gamma)^{-1}$.

BIBLIOGRAPHY

1. Sretenskii, L.N., O napravlennom izluchenii voln iz oblasti, podverzhennoi vneshnemu davleniiu (On the directional radiation of waves from a region exposed to external pressure). *PMM* Vol. 20, No. 3, 1956.

2. Voit, S.S., Volny na poverkhnosti zhidkosti, voznikaiushchie ot peremeshchaiushcheisia periodicheskoi sistemy davlenii (Waves on the surface of a fluid which arise from a traveling periodic pressure system). *PMM* Vol. 21, No. 1, 1957.

3. Voit, S.S., Volny na svobodnoi poverkhnosti i poverkhnosti razdela, voznikaiushchie ot periodicheski deistvuiushchego istochnika (Waves on the free surface and the surface of separation which arise from a periodically acting source). *Tr. Morskogo gidrofizicheskogo in-ta Akad. Nauk SSSR* Vol. 18, 1959.

4. Cherkesov, L.V., Razvitie poverkhnostnykh voln, voznikaiushchikh ot periodicheskikh peremeshchaiushchikhsia davlenii (The development of surface waves which arise from periodic traveling pressures). *Dokl. Akad. Nauk SSSR* Vol. 127, No. 4, 1959.

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